ASSIGNMENT - 2
UNIT - 2

"PARTIAL DIFFERENTATION"

Que. 1 If $u = (x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Sol. $\Rightarrow$ $$\frac{\partial^2 u}{\partial x^2} = -x\left(-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x\right) + \left[(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot (-1)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3x^2}{x^2+y^2+z^2} - 1\right)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3y^2}{x^2+y^2+z^2} - 1\right)$$

$$\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-3/2}\left(\frac{3z^2}{x^2+y^2+z^2} - 1\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left(\frac{3x^2}{x^2+y^2+z^2} + \frac{3y^2}{x^2+y^2+z^2} + \frac{3z^2}{x^2+y^2+z^2} - 3\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{3\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}} - 3\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0}$$

Hence Proved.

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Que. 2 If $u = e^{xyz}$, Prove that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Take L.H.S,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{xyz} \cdot xy\right)\right]$$

$$= \frac{\partial}{\partial x}\left[(e^{xyz} \cdot xz)xy + (e^{xyz}) \cdot x\right]$$

$$= \frac{\partial}{\partial x}\left[e^{xyz} \cdot x^2zy + e^{xyz} \cdot x\right]$$

$$= \frac{\partial}{\partial x}\left[e^{xyz}(x^2zy + x)\right]$$

$$= e^{xyz}(2zyx + 1) + \left[e^{xyz}(yz)(x^2zy + x)\right]$$

$$= e^{xyz}\left[2xyz + 1 + yz(x^2zy + x)\right]$$

$$= e^{xyz}(2xyz + 1 + x^2y^2z^2 + xyz)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Hence Proved

Que. 3 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Prove that (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 \cdot u = \frac{-9}{(x+y+z)^2}$.

(i) $$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2 - 3zx)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2 - 3xy)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}\left[3x^2+3y^2+3z^2-3xy -3yz-3zx\right]$$

$$= \frac{3}{x^3+y^3+z^3-3xyz}\left[x^2+y^2+z^2-xy-yz-zx\right]$$

[# FORMULA USED ⇒ $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$]

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}\frac{\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}}$$ Hence Proved..

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(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=$

$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$\frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$

$=\frac{-9}{(x+y+z)^2}$ Hence Proved..

Que. 4 If $x^x y^y z^z = C$ then prove that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

Sol.^{n} ⇒ Given $x^x y^y z^z = C$ —(1)

Taking logarithm on both side, we get

$x \log x + y \log y + z \log z = \log c$ —(2)

eq. (2) partially diffrentiating w.r.t x by nothing that z is the func.^{n} of x & y.

$$= \left[\log x + x \cdot \frac{1}{x}\right] + 0 + \left[\log z \cdot \frac{\partial z}{\partial x} + z \cdot \frac{1}{z}\frac{\partial z}{\partial x}\right] = 0$$

$$= [\log x + 1] + [\log z + 1]\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-[1+\log x]}{[1+\log z]}} \quad (3)$$

Similarly, $\boxed{\frac{\partial z}{\partial y} = \frac{-[1+\log y]}{[1+\log z]}} \quad (4)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial x}\left[\frac{1+\log y}{1+\log z}\right]$$

$$= -(1+\log y)\frac{\partial}{\partial x}(1+\log z)^{-1}$$

$$= -(1+\log y)(-1)(1+\log z)^{-2}\left(\frac{1}{z}\right)\frac{\partial z}{\partial x}$$

$$= \frac{1+\log y}{z(1+\log z)^2}\left\{\frac{(1+\log x)}{(1+\log z)}\right\} \text{ from eq. } (4)$$

$\boxed{\text{At } x = y = z}$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{-(1+\log x)^2}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{-1}{x(\log e + \log x)} = -(x \log ex)^{-1}$$ Hence Proved

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Que. 5 If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, then Prove that

(i) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u} \quad \text{—— } (1)$$

Differentiate w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial y \partial x} = \cos 2u \cdot 2\frac{\partial u}{\partial x}$$

$$\boxed{x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x} = (2\cos 2u - 1)\frac{\partial u}{\partial x}} \quad \text{—— } (2)$$

Diff. eq. (1) w.r.t y

$$\boxed{\frac{x\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial y}} \quad \text{—— } (3)$$

$$\frac{x^2\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u \cdot \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$\frac{x^2\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y}$$

Multiply eq. 2 by x & eq. 3 by y & Add

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial y\,\partial x} + xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\sin 2u$$

$$= 2\cos 2u \cdot \sin 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u \sin u$$

Hence Proved

Que. 6 If $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$, Verify the Euler's theoram?

Sol.^{n} ⇒ Here, $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Thus, u is a homogenous func.^{n} of x & y of degree $1/20$. $n = 1/20$

Hence to verify Euler's theoram for u we have to prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}u$

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diff. w.r.t (x)

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{-1}x^{-1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-1}x^{-1/5}\right)}{}$$

$$\frac{x\,\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{1/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

Again eq.-(2) partially diff. w.r.t (y)

$$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-\frac{4}{5}}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial u}{\partial y} = (x^{1/5}+y^{1/5})\left(\frac{1}{4}\cdot y^{-1}y^{-\frac{1}{4}}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-1}y^{-\frac{1}{5}}\right)$$

$$y\frac{du}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}y^{1/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}\right)}{(x^{1/5}+y^{1/5})^2} \quad \text{--- (4)}$$

Adding eq. (3) & (4)

$$\therefore \frac{x\,du}{dy} + \frac{y\,du}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) + (x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}\right) \left[-(x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}\right)\right]}{(x^{1/5}+y^{1/5})^2}$$

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$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4}\right) - (x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{\cancel{(x^{1/5}+y^{1/5})}(x^{1/4}+y^{1/4})\left(\frac{1}{4}-\frac{1}{5}\right)}{(x^{1/5}+y^{1/5})^{\cancel{2}}}$$

$$\boxed{x\frac{du}{dx} + y\frac{du}{dy} = \frac{[x^{1/4}+y^{1/4}]}{[x^{1/5}+y^{1/5}]}\left[\frac{1}{20}\right]} \text{ Ans.}$$

Que. 7 If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Sol.^{n} $\Rightarrow$ $\sin u = \frac{x^2+y^2}{x+y} = \frac{x^{\cancel{2}}}{\cancel{x}}\left[\frac{\left(1+\left(\frac{y}{x}\right)^2\right)}{1+\frac{y}{x}}\right]$ $\boxed{n=1}$

$f = \sin u$

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u}$$ Hence Proved

L.H.S = R.H.S

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Que. 8 Evaluate cube root 127 approximately.

Sol.^{n} ⇒

Let, $y = f(x) = \sqrt[3]{x} = x^{1/3}$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

$$\boxed{\delta y = f(x+\delta x) - f(x)}$$

$$\delta y = f'(x) \cdot \delta x$$

$$f(x+\delta x) - f(x) = f'(x) \cdot \delta x$$

Now, $\left[\because x = 125 \ \& \ \delta x = 2 \right]$

$$f(125+2) - \sqrt[3]{125} = \frac{1}{3}(125)^{-2/3}(2)$$

$$f(127) = 5 + \frac{2}{3} \times \frac{1}{25}$$

$$f(127) = 5 + \frac{2}{75}$$

$$\boxed{f(127) = 5.026666667}$$ Ans.